Locality, Causality and Noncommutative Geometry

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Abstract

We analyse the causality condition in noncommutative field theory and show that the nonlocality of noncommutative interaction leads to a modification of the light cone to the light wedge. This effect is generic for noncommutative geometry. We also check that the usual form of energy condition is violated and propose that a new form is needed in noncommutative spacetime. On reduction from light cone to light wedge, it looks like the noncommutative dimensions are effectively washed out and suggests a reformulation of noncommutative field theory in terms of lower dimensional degree of freedom. This reduction of dimensions due to noncommutative geometry could play a key role in explaining the holographic property of quantum gravity.

1 Introduction

The study of field theories in noncommutative space has attracted a great deal of attention (see for example, the review [1, 2]) after it was revealed that low-energy description of string theory in the presence of a constant NS-NS B-field background gives rise to a certain class of field theory on noncommutative space [3] with commutation relations $[x^\mu, x^\nu] = i\theta^{\mu\nu}$. One of the important problems in the study of noncommutative field theories is to understand how to define observable quantities, see for example [4, 5] for discussions about the construction of the energy-momentum tensor, and [6, 7] for the construction of gauge invariant observable in noncommutative gauge theory. In noncommutative space, locality is broken and it is not clear whether one can define local physical quantities in general.

In commutative field theories, microcausality enables us to define local observables. Also, microcausality is one of the basic assumptions leading to the notion of S-matrix.

The study of microcausality will help us understand better the notion of locality in noncommutative field theory. In noncommutative space, translational invariance is intact. However Lorentz symmetry is broken down to a smaller sub-group. This fact leads the authors of [8] to argue that the notion of a light cone is generally modified to that of a light wedge. A modified microcausality condition was proposed which state that

$$[\Phi(x), \Phi(0)]_\pm = 0, \text{if } (x^0)^2 - (\vec{x}_c)^2 < 0, \tag{1}$$

where $+(-)$ is for fermionic (or bosonic) fields and $\vec{x}_c$ includes only the commutative space direction. The relation between violation of causality and unitarity was studied in [9]. Another version where the commutator is replaced by a star-commutator has also been proposed in [10]. See also [11] where microcausality condition for composite operators has been studied. Several properties of noncommutative field theories such as CPT theorem [8, 10], spectral relations and dispersion relations of the S-matrix [11, 12] have been studied with a modified microcausality condition adapted for a light wedge. Attempts to study noncommutative field theories from an axiomatic approach have also been made [8, 10, 13].

Although the argument based on the spacetime symmetry generally allows the causal region to be enlarged to the light wedge, it is still necessary to study the causal structure in concrete models and see if and how the light cone is modified. For example, we may consider a commutative theory in $d = 6$

$$S = \int d^6x \left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + gc^{\mu\nu}\phi\partial_\mu\phi\partial_\nu\phi \right). \tag{2}$$

If we consider the case in which only $c^{45} \neq 0$, then the Lorentz symmetry is broken to $SO(3,1) \times SO(2)$. One may expect that the light wedge will arise too due to the general analysis of [8]. However as will be clear from our analysis below, microcausality condition is still given by the usual light cone for the theory (2). The crucial difference between a noncommutative theory and a generic Lorentz symmetry broken theory lies in the *nonlocality* of noncommutative interaction. This important aspect was missing in the general argument of [8].

In this paper, we study the microcausality in a concrete model, the noncommutative ϕ^3 theory. In section 2, we show that due to the phase factor characteristic of the nonplanar diagram, the light cone is modified to the light wedge in perturbative computation. Also it will be clear from our computation that this modification is generic and holds for general noncommutative field theory. In section 3 we study the energy conditions in noncommutative field theories and show that their usual form is violated. This is consistent with our result in section 2. Discussion and conclusions are made in section 4.

2   Microcausality condition in perturbative theory

2.1   Commutative case

For a vector v^μ, we use the notation $\mathbf{v}$ to denote its spatial part. We have $u \cdot v = u^0 v^0 - \mathbf{u} \cdot \mathbf{v}$. We also use the light cone coordinates. Define $v_\pm = (v^0 \pm v^5)/\sqrt{2}$ and denote the orthogonal part by $\vec{v}$. We have $u \cdot v = u_+ v_- + u_- v_+ - \vec{u} \cdot \vec{v}$.

We start with the commutative case to see how the light cone condition is derived in a canonical quantization formalism. Consider the ϕ^3 theory in six dimensions. To study the causal structure, we consider the matrix element of the field commutator

$$\mathcal{M} := \langle\alpha|[\phi_H(x_1), \phi_H(x_2)]|\beta\rangle. \tag{3}$$

Here ϕ_H is the operator in the Heisenberg picture

$$\phi_H(x) = U^\dagger(t, t_0)\phi(x)U(t, t_0), \tag{4}$$

ϕ is the field operator in the interaction picture

$$\phi(t, \vec{x}) = \int \frac{d^5\mathbf{p}}{(2\pi)^5} \frac{1}{\sqrt{2\omega_p}} \left(a_p\, e^{-ip\cdot x} + a_p^\dagger\, e^{ip\cdot x} \right)\bigg|_{x^0 = t - t_0}, \qquad \omega_p := \sqrt{\mathbf{p}^2 + m^2} \tag{5}$$

and $U(t_1, t_2)$ is the time evolution operator

$$U(t_1, t_2) = T \exp\left(ig \int_{t_1}^{t_2} dt V(t) \right), \qquad V(t) = \int d^5\vec{x}\, \frac{1}{3!} : \phi^3 : . \tag{6}$$

We use a normal ordered interaction term here. However this is inessential to our argument, the same result will be obtained without the normal ordering. We will make more comments on this at the end of this section. For simplicity, we consider the states

$$|\alpha\rangle = |\beta\rangle = U^\dagger(t_1, t_0)|0\rangle, \tag{7}$$

where $|0\rangle$ is the perturbative vacuum $a_\mathbf{p}|0\rangle = 0$. Due to translational invariance, $\mathcal{M}$ is a function of $x^\mu := x_1^\mu - x_2^\mu$.

We evaluate the matrix element (3) in perturbation theory. Up to second order in g, we have

$$\mathcal{M} = \mathcal{M}^{(0)} + \mathcal{M}^{(1)} + \mathcal{M}^{(2)} + \cdots, \tag{8}$$

where

$$\mathcal{M}^{(0)} = \langle 0|[\phi(x_1), \phi(x_2)]|0\rangle, \tag{9}$$

$$\mathcal{M}^{(1)} = ig \int_{t_1}^{t_2} ds \langle 0| \left[\phi(x_1)V(s)\phi(x_2) - \phi(x_1)\phi(x_2)V(s) + \text{c.c.} \right]|0\rangle, \tag{10}$$

$$\mathcal{M}^{(2)} = -g^2 \int_{t_1}^{t_2} ds_1 \int_{t_1}^{s_1} ds_2\, \langle 0| \quad \left[\phi(x_1)V(s_1)V(s_2)\phi(x_2) + \phi(x_1)\phi(x_2)V(s_2)V(s_1) \right.$$
$$\left. -\phi(x_1)V(s_1)\phi(x_2)V(s_2) - \text{c.c.} \right]|0\rangle. \tag{11}$$

For the ϕ^3 interaction, it is easy to see that $\mathcal{M}^{(1)} = 0$. It is easy to show that $\mathcal{M}^{(0)} = 0$ iff

$$\delta x^2 := (x_1 - x_2)^2 = 2x^+ x^- - \vec{x}^2 < 0. \tag{12}$$

For completeness we include an elementary calculation in the appendix.

For $\mathcal{M}^{(2)}$, a straightforward calculation yields

$$\mathcal{M}^{(2)} = \frac{-g^2}{(2\pi)^{10}} \int \frac{d^5\mathbf{q}_2 d^5\mathbf{q}_3}{2\omega_{q_2} 2\omega_{q_3}(2\omega_{p_1})^2} e^{-ip_1 x} f - \text{c.c.}, \tag{13}$$

where

$$\mathbf{p}_1 = \mathbf{q}_2 + \mathbf{q}_3, \qquad p_1^0 = \omega_{p_1} = \sqrt{(\mathbf{q}_2 + \mathbf{q}_3)^2 + m^2}, \tag{14}$$

$$f = \frac{t}{2i\tilde{\omega}} + \frac{1}{2\tilde{\omega}^2} + \frac{1}{\tilde{\omega}\hat{\omega}} - \frac{1}{\tilde{\omega}\omega_{p_1}} + \frac{1}{\tilde{\omega}\hat{\omega}} e^{i\tilde{\omega}t} - \frac{1}{2\tilde{\omega}^2} e^{-i\tilde{\omega}t}, \tag{15}$$
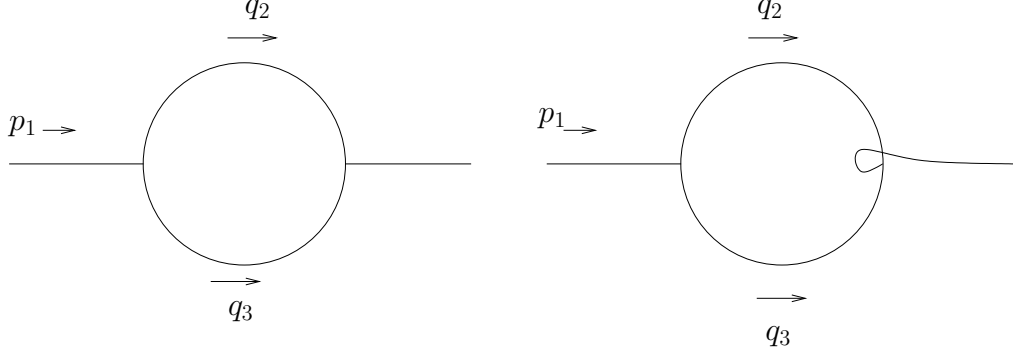
Figure 1: Planar and nonplanar contributions

$$\hat{\omega} := \omega_{q_2} + \omega_{q_3} + \omega_{p_1}, \tag{16}$$

$$\tilde{\omega} := -\omega_{q_2} - \omega_{q_3} + \omega_{p_1}. \tag{17}$$

The contributions leading to (13) are given in figure 1, the planar diagram.

Using

$$\int \frac{d^5 \mathbf{q}}{\omega_q} = \int_0^\infty \frac{dq^+}{q^+} d^4 \vec{q} \bigg|_{q^- = (\vec{q}^2 + m^2)/2q^+}, \tag{18}$$

we obtain the light cone representation of $\mathcal{M}^{(2)}$ as

$$\mathcal{M}^{(2)} = g^2 \left[\int_0^\infty dq_2^+ \int_0^\infty dq_3^+ G(q_i^+, x) - \int_0^\infty dq_2^+ \int_0^\infty dq_3^+ G(q_i^+, -x) \right], \tag{19}$$

where

$$G(q_i^+, x) := e^{-i\frac{\delta x^2}{2x^+}(q_2^+ + q_3^+) - i\frac{m^2 x^+}{2}\left(\frac{1}{q_2^+} + \frac{1}{q_3^+}\right)} H(q_i^+, x) \tag{20}$$

and the kernel $H(q_i^+, x)$ is defined as

$$H(q_i^+, x) := \int \frac{d^4 \vec{q}_2 d^4 \vec{q}_3}{(2\pi)^{10}} e^{-i\frac{x^+}{2q_2^+}\left(\vec{q}_2 - q_2^+ \frac{\vec{x}}{x^+}\right)^2} e^{-i\frac{x^+}{2q_3^+}\left(\vec{q}_3 - q_3^+ \frac{\vec{x}}{x^+}\right)^2} \frac{f e^{-i\tilde{\omega} t}}{16 q_2^+ q_3^+ \omega_{p_1}^2}. \tag{21}$$

The frequencies ω_{q_i} and ω_{p_1} are to be written in terms of the light cone and transverse components. We have

$$\omega_{q_i} = \frac{1}{\sqrt{2}}(q_i^+ + \frac{1}{2q_i^+}(\vec{q}_i^2 + m^2)), \quad i = 2, 3, \tag{22}$$

$$\omega_{p_1}^2 = (\vec{q}_2 + \vec{q}_3)^2 + m^2 + \frac{1}{2}\left[q_2^+ + q_3^+ - \frac{1}{2q_2^+}(\vec{q}_2^2 + m^2) - \frac{1}{2q_3^+}(\vec{q}_3^2 + m^2)\right]^2. \tag{23}$$

In the above we have chosen to write G in the form (20) which will be convenient for our analysis. Note that the exponential part in (20) is kinematical and is independent of the interaction. The details of dynamics of the theory enter through $H(q_i^+, x)$ from the last piece in (21).

The convergence properties of the q_2^+, q_3^+ integrals in (19) depend crucially on the sign of δx^2. For large q_i^+,

$$\omega_{q_i} \sim q_i^+, \quad \omega_{p_1} \sim q_2^+ + q_3^+, \quad \tilde{\omega} \sim 1/q_i^+, \quad \hat{\omega} \sim q_2^+ + q_3^+, \tag{24}$$

up to numerical proportional constants. It follows that f and $H(q_i^+, x)$ grow at most polynomially with q_i^+. The large q_i^+ behaviour of $G(q_i^+, x)$ is thus dominated by the exponential part,

$$G(q_i^+, x) \sim e^{-i(q_2^+ + q_3^+)\frac{\delta x^2}{2x^+}}, \quad \text{for large } q_i^+. \tag{25}$$

Without loss of generality [1], we assume $x^+ > 0$, $G(q_i^+, x)$ thus vanishes exponentially at large q_i^+ on the upper half-plane if $\delta x^2 < 0$. It can also be verified that there is no pole enclosed when the integration path is rotated. Thus we can rotate the contours of the integration to the imaginary axes as follows without getting any extra contribution:

$$q_i^+ \to i q_i^+ \quad \text{in the first integral in (19)},$$
$$q_i^+ \to -i q_i^+ \quad \text{in the second integral in (19)}. \tag{27}$$

And we obtain

$$\mathcal{M}^{(2)} = g^2 \left[\int_0^\infty dq_2^+ \int_0^\infty dq_3^+ G(iq_i^+, x) - \int_0^\infty dq_2^+ \int_0^\infty dq_3^+ G(-iq_i^+, -x) \right]. \tag{28}$$

Since the frequencies $\omega_{q_i}, \omega_{p_1}$ change sign under $q^+ \to -q^+$ [2], $G(-iq_i^+, -x) = G(iq_i^+, x)$ and hence the two terms in (19) are identical:

$$\mathcal{M}^{(2)} = \quad -\int_0^\infty dq_2^+ \int_0^\infty dq_3^+ e^{(q_2^+ + q_3^+)\frac{\delta x^2}{2x^+} - \frac{m^2 x^+}{2}\left(\frac{1}{q_2^+} + \frac{1}{q_3^+}\right)} H(iq_i^+, x)$$
$$+ \quad \text{same term as above.} \tag{29}$$

Therefore $\mathcal{M}^{(2)}$ vanishes if the first term of (29) is finite, i.e.,

$$\int_0^\infty dq_2^+ \int_0^\infty dq_3^+ e^{(q_2^+ + q_3^+)\frac{\delta x^2}{2x^+} - \frac{m^2 x^+}{2}\left(\frac{1}{q_2^+} + \frac{1}{q_3^+}\right)} H(iq_i^+, x) < \infty. \tag{30}$$

As we showed above, $H(q_i^+, x)$ grows at most polynomially with q_i^+, thus the integral converges at large q_i^+ if $\delta x^2 < 0$. For small q_i^+, we have the damping factor ($m^2 > 0$ since the theory is not tachyonic) in the exponent, thus the integral converges also at $q_i^+ = 0$. Therefore we obtain the result that, up to order g^2, the light cone is not modified by interaction.

We remark that at the tree level, the matrix element takes a similar form. See (69). In fact the form (19) and (20) is quite general and universal. In general, the matrix element at a certain order of coupling takes the form

$$\mathcal{M}^{(n)} \approx g^n \left[\int \prod_i dq_i^+ G(q_i^+, x) - \int \prod_i dq_i^+ G(q_i^+, -x) \right], \tag{31}$$

where

$$G(q_i^+, x) := \exp\left(-i\frac{\delta x^2}{2x^+}\sum_{i\in I} q_i^+ - i\frac{m^2 x^+}{2}\sum_{i\in I} 1/q_i^+\right) H(q_i^+, x) \tag{32}$$

---

[1] If $x^+ < 0$, one can consider $[\phi(x_2), \phi(x_1)]$. Or if one want, one can keep $x^+ < 0$ for the analysis. All one need to do is to perform an opposite rotation of contours to the one in (27):

$$q_i^+ \to -i q_i^+ \quad \text{in the first integral in (19)}, \quad q_i^+ \to i q_i^+ \quad \text{in the second integral in (19)}, \tag{26}$$

and the same analysis leads to the light cone condition $\delta x^2 < 0$.

[2] Although it does not follow immediately from (23), one should change the sign of all frequencies simultaneously under $q^+ \to -q^+$ for analytically.

and $H(q_i^+, x)$ is some kernel which contains the details of the interaction. The momentum integration ranges over all the independent momenta after imposing momentum conservation at each vertex. The index i in the sum ranges over a subset I of them. For example, for the diagram in figure 2, the independent momenta to be integrated can be chosen to be $\mathbf{q}_2, \mathbf{q}_3 \cdots, \mathbf{q}_n$ and the set I may be taken to be $\{2, 3\}$. A similar argument can be applied and one arrives at the same light cone. The generalization to theory with more couplings and masses is straightforward.
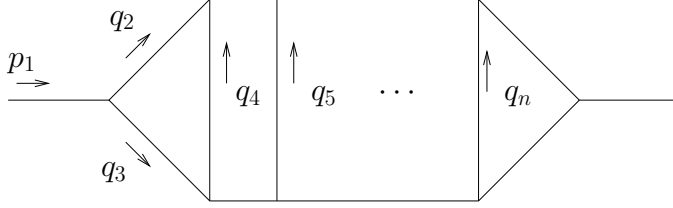

Figure 2: Higher order contributions

2.2 Noncommutative case

We now turn to the noncommutative case. Since theory with noncommutativity in time has problem with unitarity [14], see also [9,15], we will consider spatial noncommutativity We can choose the light cone direction to be commutative.

$$\left[x^i, x^j\right] = i\theta^{ij}, \quad i = 1, 2, 3, 4, \tag{33}$$

$$\theta^{\pm i} = \theta^{+-} = 0. \tag{34}$$

In noncommutative case, we can classify the terms contributing to $\mathcal{M}^{(2)}$ into planar and nonplanar parts. The planar parts give exactly the same terms (apart from numerical coefficient) as the commutative case. For the nonplanar parts, we have the additional phase factor

$$P(q_2, q_3) = e^{i\vec{q}_2\theta\vec{q}_3}. \tag{35}$$

Consider the contribution coming from non-planar part. Up to a numerical coefficient,

$$\mathcal{M}_{\mathrm{np}}^{(2)} \sim \int_0^\infty dq_2^+ \int_0^\infty dq_3^+ \left[G_{\mathrm{np}}(q_i^+, x) - G_{\mathrm{np}}(q_i^+, -x) \right], \tag{36}$$

where

$$G_{\mathrm{np}}(q_i^+, x) := e^{-i\frac{\delta x^2}{2x^+}(q_2^+ + q_3^+) - i\frac{m^2 x^+}{2}\left(\frac{1}{q_2^+} + \frac{1}{q_3^+}\right)} H_{\mathrm{np}}(q_i^+, x) \tag{37}$$

and the kernel H_{np} is

$$H_{\mathrm{np}}(q_i^+, x) := \int \frac{d^4\vec{q}_2 d^4\vec{q}_3}{(2\pi)^{10}} \, e^{-i\frac{x^+}{2q_2^+}\left(\vec{q}_2 - q_2^+\frac{\vec{x}}{x^+}\right)^2} e^{-i\frac{x^+}{2q_3^+}\left(\vec{q}_3 - q_3^+\frac{\vec{x}}{x^+}\right)^2} \frac{f e^{-i\tilde\omega t}}{16 q_2^+ q_3^+ \omega_{p_1}^2} P(q_2, q_3), \tag{38}$$

where f is defined by (15) above.

To see the possibility of contour deformation, we examine the large q_i^+ behaviour of $H_{\mathrm{np}}(q_i^+, x)$. The $\vec{q}$-integral in H_{np} is of the form

$$H_{\mathrm{np}}(q_i^+, x) = \int \frac{d^4\vec{q}_2 d^4\vec{q}_3}{(2\pi)^{10}} e^{A_2(\vec{q}_2 - \vec{a}_2)^2 + A_3(\vec{q}_3 - \vec{a}_3)^2 + i\vec{q}_2\theta\vec{q}_3} \frac{f e^{-i\tilde\omega t}}{16 q_2^+ q_3^+ \omega_{p_1}^2}, \tag{39}$$

where $A_2 = -\frac{ix^+}{2q_2^+}$, $A_3 = -\frac{ix^+}{2q_3^+}$, $\vec{a}_2 = \frac{q_2^+\vec{x}}{x^+}$ and $\vec{a}_3 = \frac{q_3^+\vec{x}}{x^+}$. As checked above, $fe^{-i\tilde{\omega}t}/16q_2^+q_3^+\omega_{p_1}^2$ diverges at large q_i^+ at most polynomially. Thus we have for large q_i^+,

$$H_{\mathrm{np}} \sim \int d^4\vec{q}_2 d^4\vec{q}_3 e^{A_2(\vec{q}_2-\vec{a}_2)^2+A_3(\vec{q}_3-\vec{a}_3)^2+i\vec{q}_2\theta\vec{q}_3} \tag{40}$$

$$= \int d^4\vec{q}_2 e^{A_2(\vec{q}_2-\vec{a}_2)^2} \int d^4\vec{q}_3 \exp\left(A_3\left(\vec{q}_3 + \frac{i\overrightarrow{q_2\theta}-2A_3\vec{a}_3}{2A_3}\right)^2 + \frac{(\overrightarrow{q_2\theta})^2 + 4iA_3\vec{q}_2\theta\vec{a}_3}{4A_3}\right). \tag{41}$$

Now we choose θ^{ij} to be in the canonical form

$$\theta^{ij} = \begin{pmatrix} (\theta')_{2\times 2} & \\ & (\theta'')_{2\times 2} \end{pmatrix} = \begin{pmatrix} & \theta' & & \\ -\theta' & & & \\ & & & \theta'' \\ & & -\theta'' & \end{pmatrix}, \tag{42}$$

then

$$(\overrightarrow{q_2\theta})^2 = \theta'^2(\vec{q}_2')^2 + \theta''^2(\vec{q}_2'')^2, \tag{43}$$

where $\vec{q}_2'$ and $\vec{q}_2''$ represent the $1,2$ components and $3,4$ components of $\vec{q}_2$ respectively. The integral (41) factorizes into a product of 2 dimensional integrals. Integrating out $\vec{q}_3$, we obtain

$$H_{\mathrm{np}} \sim \int d^2\vec{q}_2' \exp\left[(\vec{q}_2')^2\left(A_2 + \frac{\theta'^2}{4A_3}\right) - \vec{q}_2' \cdot \left(2A_2\vec{a}_2 - i\overrightarrow{\theta'a_3}\right) + A_2\vec{a}_2^2\right]$$

$$\times \int d^2\vec{q}_2'' \exp\left[\,\theta' \text{ replaced by } \theta''\,\right]. \tag{44}$$

This can be easily integrated and it gives

$$H_{\mathrm{np}} \sim e^{F'}e^{F''}, \tag{45}$$

where

$$F' := \frac{1}{E'}\left[\frac{\theta'^2 A_2}{4A_3}\vec{a}_2^2 + iA_2\vec{a}_2\overrightarrow{\theta'a_3} + \frac{(\overrightarrow{\theta'a_3})^2}{4}\right], \quad \text{and} \quad E' := A_2 + \frac{\theta'^2}{4A_3} \tag{46}$$

and a similar expression for F'' with θ' replaced by θ''. Substitute the expressions for $A_2, A_3, \vec{a}_2, \vec{a}_3$, we obtain

$$F' = -i\frac{\theta'^2\vec{x}'^2 q_2^+ q_3^+(q_2^+ + q_3^+)}{2x^+(-x^{+2} + \theta'^2 q_2^+ q_3^+)} \tag{47}$$

whose large q_i^+ behaviour can be read off easily

$$F' \to -\frac{i\vec{x}''^2}{2x^+}q_2^+ \text{ as } q_2^+ \to \infty, \tag{48}$$

$$F' \to -\frac{i\vec{x}''^2}{2x^+}q_3^+ \text{ as } q_3^+ \to \infty. \tag{49}$$

Note that θ'-dependence is cancelled completely! Similarly, we have the same results for F''. Thus for large q_i^+, the integral factor $H_{\mathrm{np}}(q_i^+, x)$ contributes

$$H_{\mathrm{np}}(q_i^+, x) \sim e^{-i\frac{\vec{x}_{nc}^2}{2x^+}(q_2^+ + q_3^+)} \tag{50}$$

for each subspace of $\vec{x}_{nc}$ with non-zero noncommutativity. This is in contrast to the commutative case where the dependence on q_i^+ is subdominant compared to that of $G(q_i^+, x)$. Because of (50), the large q_i^+ behaviour of $G_{np}(q_i^+, x)$ will be modified. Notice that the large q_i^+ behaviour of H_{np} matches the exponential factor of (37), we have

$$
\mathcal{M}^{(2)} = \text{coeff.} \times \int_0^\infty dq_2^+ \int_0^\infty dq_3^+ \left[e^{-i(q_2^+ + q_3^+)x^- - i\frac{m^2 x^+}{2}\left(\frac{1}{q_2^+} + \frac{1}{q_3^+}\right) + i\frac{\vec{x}^2}{2x^+}(q_2^+ + q_3^+)} H_{np}(q_i^+, x) - \text{c.c.} \right]
$$

$$
\sim \int_0^\infty dq_2^+ \int_0^\infty dq_3^+ \left[e^{-i(q_2^+ + q_3^+)\frac{(\Delta x)^2}{2x^+}} - \text{c.c.} \right], \tag{51}
$$

where

$$
(\Delta x)^2 := 2x^+ x^- - \vec{x}_c^2. \tag{52}
$$

$\vec{x}_c$ represents the commutative subspace of $\vec{x}$. In the second line of (51), we have omitted factors which do not exponentially diverge at large q_i^+.

Now we can repeat the same argument as in the commutative case:

$$
\text{We can rotate the contour if } (\Delta x)^2 < 0;
$$

$$
\text{Integrals in (51) converge and cancel out if } (\Delta x)^2 < 0. \tag{53}
$$

Thus we see that the light cone $((\delta x)^2 < 0)$ is modified to light wedge $((\Delta x)^2 < 0)$ in order for the commutator to be zero.
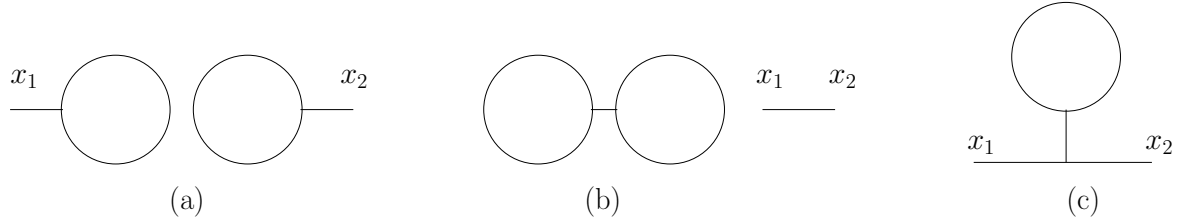

Figure 3: Unimportant contributions

A couple of remarks are in order.

1. In the above, we have done the computation using a normal ordered vertex. However this point is inessential. If we use a vertex without normal ordering, then we will get additional terms as shown in figure 3. These diagrams arise from the self-contractions of the vertex and will give additional contributions to $\mathcal{M}^{(2)}$. However it is easy to see that there will not be any noncommutative phase factor involved for these terms. Because of this, they will not modify the microcausality condition. The modification to the microcausality condition arises solely from the nonplanar diagrams in figure 1.

2. Another effect of nonlocality in noncommutative field theory is the emergence of IR singularity from integrating out UV degree of freedoms. This mixing of scales is called IR/UV mixing phenomena [17, 18]. IR/UV mixing implies a breakdown of the standard Wilsonian effective description. IR/UV mixing may also results in instability of the perturbative vacuum. This happens for example for the case of noncommutative QED [19] and for the ϕ^3 theory [17] we considered. [3] Since the

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[3]The ϕ^3 theory suffers also from the usual instability that it does not has a stable vacuum. However this is a separate issue.

vacuum is unstable, in principle our perturbative analysis is not valid. However our analysis can be straightforwardly extended to other theory where this is not a problem, for example ϕ^4 in 4-dimension. It is clear from our analysis above that the modification of the light cone to light wedge is due to the inclusion of the noncommutative phase factor P which appears in (38) for the nonplanar diagram. For the ϕ^4 theory, there will be a different f. However the precise form of f is unimportant for our argument. Our analysis above go through. Therefore we conclude that the modification from light cone to light wedge is quite general and is valid even if there is IR/UV mixing so long as the IR/UV mixing effect does not result in instability of the vacuum.

3. Obviously supersymmetrizing the theory will not change our result. Again having more particles circulating in the loop will only modify the form of the function f, which is not important for our argument.

4. Using the equations (31) and (32), our analysis can be readily generalized to higher order. In general a noncommutative phase factor will appear for the nonplanar diagrams. For those nonplanar diagrams which has a phase factor which intertwines the momenta in the set I in (32), our analysis can be applied straightforwardly and the light wedge is obtained. For example for the planar contribution in figure 2, we have the noncommutative counterpart in figure 4 which give rise to the light wedge due to the phase factor $e^{i\vec{q}_2\theta\vec{q}_3}$ associated with the diagram.

5. Theory with lightlike noncommutativity is unitary perturbatively [16] and presents another interesting case. For the case say $\theta := \theta^{i-} \neq 0$, one quantize the theory using the light cone time x^+ as the time coordinate. Since $\theta^{i+} = 0$, the theory is local in the light cone time and the operator formalism is well defined. Our analysis can be applied directly and we find

$$H_{\mathrm{np}}(q_i^+, x) \sim e^{i\frac{\theta^2}{2x^+}q_2^+q_3^+(q_2^+ + q_3^+)}. \tag{54}$$

This implies that the commutator vanishes only for $x_1^+ = x_2^+$. This agrees precisely with the Galilean spacetime symmetry which is in residual in the presence of $\theta^{i-} \neq 0$.
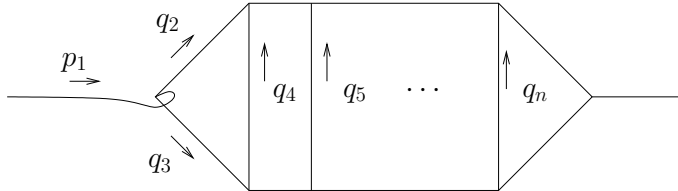

Figure 4: Higher order nonplanar contributions

In conclusion, our analysis and result is applicable to general noncommutative theories, so long as the theory admits nonplanar diagrams and the perturbative vacuum is not instabilized by the IR/UV mixing effect.

3 Energy condition in noncommutative geometry

Another way to study the causality issue is to consider energy condition. For review, see [20], also [21]. For applications in string theory, see for example [22]. Let us first consider the dominant energy condition in details. The discussion for the other energy conditions is similar. An energy-momentum tensor $T_{\mu\nu}$ is said to satisfy the dominant energy condition if for every future directed time like vector t^μ, $T_{\mu\nu}t^\nu$ is future directed, time like or null. One can show that when it is satisfied, the velocity of the energy-flow cannot exceed the speed of light.

To warm up, let us first consider an example in commutative theory

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi) \tag{55}$$

with $V(\phi) = \frac{1}{2}m^2\phi^2 + V_{\text{int}}$. The energy-momentum tensor is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}. \tag{56}$$

We now show that the dominant energy condition is satisfied if $V > 0$. Let t^μ be timelike: $t^\mu t_\mu > 0$ and future directed: $t^0 > 0$. Then

$$u_\mu := T_{\mu\nu}t^\nu = \partial_\mu\phi(\partial\phi \cdot t) - t_\mu\mathcal{L} \tag{57}$$

and

$$u_\mu u^\mu = (\partial\phi)^2 V(\phi) + t^2\mathcal{L}^2. \tag{58}$$

Thus $u_\mu u^\mu \geq 0$ if $V(\phi) > 0$. For example, if $m^2 > 0$ and $V_{\text{int}} > 0$. As for u_0, we have

$$u_0 = \left[\frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\partial_i\phi)^2 + V\right]t^0 + \partial_0\phi\partial_i t^i. \tag{59}$$

t future directed and timelike implies $t^0 > |\mathbf{t}|$. If $V > 0$ also, then

$$u_0 > |\mathbf{t}|\left[\frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\partial_i\phi)^2\right] + \partial_0\phi\partial_i t^i > |\partial_0\phi|\left[|\mathbf{t}||\partial_i\phi| \pm t^i\partial_i\phi\right] \geq 0 \tag{60}$$

is future directed.

Now consider the noncommutative ϕ^4 theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi * \phi * \phi * \phi. \tag{61}$$

Alternatively, we can also consider the Lagrangian which gives the same action

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi * \partial^\nu\phi - \frac{1}{2}m^2\phi * \phi - \frac{g}{4!}\phi * \phi * \phi * \phi. \tag{62}$$

First, we need to construct the symmetric and conserved energy-momentum tensor. This problem has been considered in [5] and the authors found that they cannot be satisfied simultaneously. Here we show that one can construct such an energy-momentum tensor if one allow it to be path-dependent. This is also the case in noncommutative Yang-Mills theory [4]. We define the energy momentum tensor

$$\hat{T}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\int_C dy^\lambda\,\partial_\lambda\phi(y)(\phi * \phi * \phi)(y)\right]. \tag{63}$$

Here the integration is carried out along an arbitrary path C connecting from a reference point x_0 to x. Obviously, $\hat{T}_{\mu\nu} = \hat{T}_{\nu\mu}$ and also $\hat{T}_{\mu\nu}$ is real. Using equation of motion, it is easy to verify that

$$\partial^\mu \hat{T}_{\mu\nu} = 0. \tag{64}$$

For the alternative choice of the Lagrangian (62), we can consider the path-dependent energy-momentum tensor

$$\hat{T}_{\mu\nu} = \frac{1}{2}\partial_\mu\phi * \partial_\nu\phi + \frac{1}{2}\partial_\nu\phi * \partial_\mu\phi - g_{\mu\nu}\left[\frac{1}{2}\partial_\lambda\phi * \partial^\lambda\phi - \frac{1}{2}m^2\phi * \phi\right.$$
$$\left. - \frac{g}{3!2}\int_C dy^\lambda \left((\partial_\lambda\phi * \phi * \phi * \phi)(y) + (\phi * \phi * \phi * \partial_\lambda\phi)(y)\right)\right]. \tag{65}$$

Again, it is easy to show that $\hat{T}_{\mu\nu}$ is real, symmetric and conserved.

Let us now check the dominant energy condition in the noncommutative case. Note that (63) is of the same form as (56). Thus we arrive at (58) and (59) again. However now the analogy of V_{int} is given by the path-dependent terms in (63) and (65) and they are not positive definite for non-zero noncommutativity. One can also show that (65) does not satisfy the dominant energy condition. It is straightforward to generalize the argument for other noncommutative field theories. Therefore we conclude that the dominant energy condition is generally violated in noncommutative theory. It is also easy to verify that none of the standard form of energy condition (e.g. null, strong, weak) are satisfied. On the other hand, if we naively integrating out the whole noncommutative subspace, then energy conditions are satisfied. This is reminiscent of the averaged energy condition [23].

Given that the causal structure of the noncommutative theory is modified, it is not surprising that the usual form of energy condition is violated. The real challenge and really interesting problem is to derive the appropriate generalization of the energy condition for noncommutative spacetime.

4 Discussion

In this paper, we have shown that the causality condition in noncommutative field theory is generally modified from a light cone to a light wedge. The phase factor $e^{iq_2\theta q_3}$ in (41), which arises due to the nonlocal nature of the noncommutative interaction, is crucial to modify the large q_i^+ behaviour of the kernel $H_{\text{np}}(q_i^+, x)$ and hence the modified causality condition. In contrast, a local Lorentz symmetry breaking interaction will generally not modify the light cone. Nonlocality of interaction plays the key role. See also [24] for other considerations on how noncommutative geometry may modify the causality condition.

Based on the modified dispersion relation in the noncommutative theory at nonzero temperature, the authors of [25] show that the low momenta modes have superluminous group velocity. The authors of [26] constructed and studied the classical dynamics of solitons in noncommutative gauge theories. They found that these solitons can travel with speed faster than light (with no upper bound in the noncommutative direction). However as these authors argued, since one cannot send information backward in time, so this does not violate causality. One may also deduce the light wedge in this setting. Our result of having the light cone changed to the light wedge is consistent with their result. However the analysis and result here is more general since it applies without the need of specific modified dispersion relation nor of the existence and the construction of solitons with specific properties.

In [10], a new form of Wightman functions, which are defined as vacuum expectation value of star-product of fields, have been considered. The motivation was that the new form of Wightman functions contain the noncommutativity explicitly. On the other hand, as the authors argued, on applying the reconstruction theorem the usual Wightman functions will lead to commutative field theory (which are invariant under the smaller Lorentz group), but not to noncommutative field theory since there is no trace of the noncommutativity parameter θ in the usual Wightman function. We remark that this is not the case. The Wightman functions can depend on θ just as they depends on the coupling constants. This can be easily demonstrated in perturbation theory, as is clear from our calculations. Also the authors proposed another form of microcausality condition where the star-commutator of fields vanishes at separation which is spacelike with respect to the light wedge. We note that, due to momentum conservation, it is

$$\langle\alpha|[\phi(x_1),\phi(x_2)]|\beta\rangle = \langle\alpha|[\phi(x_1),\phi(x_2)]_*|\beta\rangle, \tag{66}$$

for states $|\alpha\rangle, |\beta\rangle$ with the same momentum. Thus as far as matrix elements of the above type are of concern, there is no difference in considering the standard commutator or the star-commutator. The usual commutator is smart enough to know about the modified causality condition in noncommutative geometry.

As we explained, the emergence of the modified causality condition is characteristic of noncommutative geometry and is generic and independent of the details of the type of noncommutative interaction. One can expect that this drastic change of the notion of causality to be a rather clear phenomenological signal for noncommutative geometry. See for example [27–29] for some discussions of the phenomenological aspect of noncommutative field theory. If the universe was noncommutative at the early stage, region that one would traditionally taken to be not in causal contact (outside the light cone) may indeed be causally related according to the light wedge. This may offer an alternative scenario to inflation for explaining the horizon problem of the universe. For other applications of noncommutative geometry to cosmology, see for example [30].

In this paper, we have seen that nonlocality of noncommutative geometry leads to a modification of the light cone. String theory is another nonlocal theory. The effect of interaction on the string light cone [31] has been studied in [32] for the case of flat string theory and [33] for the pp-wave string theory. The result is that the light cone is modified in the flat case, but not in the pp-wave case [33]. This is related to the fact that the pp-wave 3-string interaction is more localized compared to the flat space 3-string interaction. It is important to identify and understand better the nonlocal effects in string theory. This should help us to understand better the nature of the theory of quantum gravity, which is believed to be nonlocal also.

The modification of the light cone in noncommutative field theory is intriguing. As far as causality is of concern, no trace of the noncommutative coordinates enter. Our result suggests that a reformulation of the noncommutative theory in terms of one living in lower dimensional spacetime maybe able to capture the causal aspects of the physics more accurately. This possibility of formulation in terms of lower dimensional degree of freedom reminds us of the holographic property of quantum gravity [34]. The dynamical reduction of spacetime dimensions at short distance scale is also seen in the very interesting recent works [35]. Since quantum gravity effects is believed to quantize the spacetime and make it noncommutative, noncommutativity geometry may play a key role in explaining holography. See also [36, 37] for more considerations in support of this view.

Acknowledgements

A   Light cone for free theory

Consider real scalar field in $(d+1)$-dimensions. It is easy to calculate that

$$[\phi(x), \phi(0)] = D(x) - D(-x), \quad D(x) := \int \frac{d^d \mathbf{q}}{(2\pi)^d} \frac{1}{2\omega_q} e^{-iq\cdot x} \tag{67}$$

Using (18), it is easy to rewrite D as

$$D(x) = \int_0^\infty dq^+ G(q^+, x), \quad G(q^+, x) := \frac{1}{4\pi q^+} \int \frac{d^{d-1}\vec{q}}{(2\pi)^{d-1}} e^{-iq\cdot x}. \tag{68}$$

The kernel $G(q^+, x)$ can be easily evaluated and we obtain

$$[\phi(x), \phi(0)] = \int_0^\infty dq^+ [G(q^+, x) - G(q^+, -x)], \tag{69}$$

$$G(q^+, x) = \frac{1}{4\pi q^+} \left(\frac{iq^+}{2\pi x^+} \right)^{\frac{d-1}{2}} \exp \left[-iq^+ \frac{\delta x^2}{2x^+} - \frac{im^2 x^+}{2q^+} \right]. \tag{70}$$

The convergence properties of the q^+ integral depends crucially on the sign of $\delta x^2 = 2x^+ x^- - \vec{x}^2$. Without loss of generality, we assume $x^+ > 0$. If $\delta x^2 < 0$, then we can rotate the contours in (69) as follows without getting any extra contribution:

$$q^+ \to iq^+ \quad \text{in the first integral,}$$
$$q^+ \to -iq^+ \quad \text{in the second integral.} \tag{71}$$

Moreover since

$$G(iq^+, x) \sim e^{q^+ \frac{\delta x^2}{2x^+} - \frac{m^2 x^+}{2q^+}} \tag{72}$$

the integral converges for $q^+ \to 0$ and $q^+ \to \infty$ if $\delta x^2 < 0$. Hence the two terms in (69) cancels and the commutation vanishes if $\delta x^2 < 0$. On the other hand, if $\delta x^2 > 0$, we may perform the opposite contour rotation. The integral converges for $q^+ \to \infty$ but not for $q^+ \to 0$. Therefore we obtain that the commutator vanishes iff $\delta x^2 < 0$.

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